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A MODEL FOR VISCOPLASTIC MATERIALS WITH TEMPERATURE DEPENDENCE

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Abstract—This paper extends an isothermal viscoplastic model proposed by the authors to include temperature dependence. The underlying hypotheses are a superposition of viscoelastic and plastic response in conjunction with thermorheologically simple behavior with temperature. Techniques are described for using experimental data to determine the functions that define the model and these techniques are applied to one set of such data.

1. ISOTHERMAL THEORY

We begin with a discussion of the isothermal theory introduced in [1], in which the underlying hypothesis is a superposition of viscoelastic and plastic behavior. As in that paper, we restrict ourselves to a one-dimensional theory and to essentially compressive loadings, or, more precisely, to situations in which the material yields only in compression;† this allows the use of a nonstandard description of plasticity in which the basic constitutive equation is in *integrated* form.

1.1. *The plastic portion of the model*

This portion of the model is based on the constitutive equation

$$\varepsilon = F(\pi, y), \tag{1}$$

where ε is the (compressive) **strain**, π the (compressive) **stress**, and y the **yield stress**. Here

$$\pi \leqslant y, \quad y_0 \leqslant y, \tag{2}$$

with y_0 the **initial yield-stress** and

$$F(0, y_0) = 0. \tag{3}$$

This constitutive equation is displayed graphically in Fig. 1.

The ε–y curve given by $\varepsilon = F(y, y)$ is the **yield envelope**. This curve is eventually traversed by loading processes with increasing stress, and, because of its importance, we give it the special designation

$$Y(y) = F(y, y), \quad y \geqslant y_0. \tag{4}$$

For fixed y the ε–π curve $\varepsilon = F(\pi, y)$ represents elastic loading and unloading within the yield envelope. We assume that each of these curves is linear, so that the corresponding (elastic) **compliance**

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† The materials we discuss are rarely used in tension, where they are typically quite weak. Our theory is easily placed in a more classical framework which allows yielding also in tension.

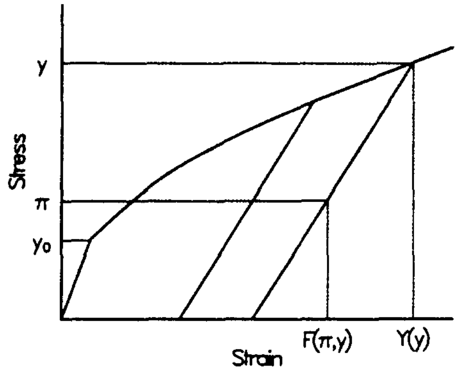

Fig. 1. The plastic portion of the model.

$$C(y) \equiv \partial F(\pi, y)/\partial \pi \tag{5}$$

is independent of π. For each y,

$$\varepsilon = F(0, y) \tag{6}$$

represents the permanent strain corresponding to y; we denote this strain by

$$p(y) \equiv F(0, y). \tag{7}$$

Finally, we write

$$M(\pi, y) \equiv \partial F(\pi, y)/\partial y. \tag{8}$$

We assume throughout that

$$C(y) > 0, \quad M(\pi, y) > 0. \tag{9}$$

The second part of (9) is the requirement that $F(\pi, y)$ behave as shown in Fig. 2; it ensures that the elastic loading and unloading curves do not cross. Note that, by (4), (5), (8), and (9), $Y'(y)$ always exceeds the corresponding elastic compliance: $Y'(y) > C(y)$.

We now consider processes in which $\varepsilon(t)$, $\pi(t)$ and $y(t)$ are functions of the time t. We assume that the material is in its virgin state at times prior to $t = 0$:

$$\pi(t) = 0, \quad y(t) = y_0, \quad \varepsilon(t) = 0 \quad \text{for} \quad t \leqslant 0, \tag{10}$$

an assumption consistent with (3).

Our final step is to prescribe a flow rule governing the evolution of $y(t)$ with t. Since

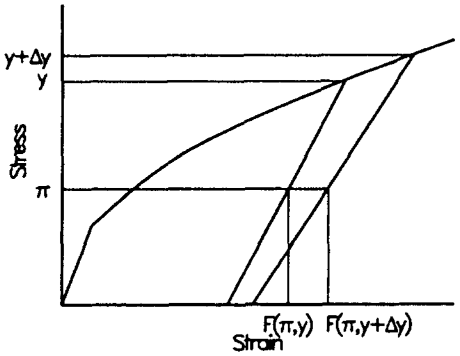

Fig. 2. Graphical description of the assumption $M(\pi, y) > 0$.

we consider yielding only in compression, $\pi(t) \leqslant y(t)$ with $y(t)$ nondecreasing. Moreover $y(t)$ increases when and only when $\pi(t) = y(t)$ and $\pi(t)$ is increasing, in which case $\pi(t)$ increases with $y(t)$. These restrictions are summarized in the **flow rule**:

$$\dot{y} = \begin{cases} \max\{\dot{\pi}, 0\} & \text{if } \pi = y, \\ 0 & \text{if } \pi < y. \end{cases} \tag{11}$$

Let $\pi_m(t)$ denote the past maximum of stress:

$$\pi_m(t) = \max\{\pi(s) : 0 \leqslant s \leqslant t\}. \tag{12}$$

Because we allow yielding only in compression, the yield condition (11) admits an alternative expression in terms of π_m:

$$y(t) = \max\{y_0, \pi_m(t)\}. \tag{13}$$

Generally this will not be true when we allow the temperature to vary.

We now derive some well known consequences of this constitutive model. These results will be useful when we generalize the model to *nonisothermal* processes.

Using (5) and (8), we can write (1) as a rate law

$$\dot{\varepsilon} = C(y)\dot{\pi} + M(\pi, y)\dot{y}. \tag{14}$$

Since C is independent of π, from the integrability condition

$$\partial C/\partial y = \partial M/\partial \pi, \tag{15}$$

we deduce the existence of a function $R(y)$ (independent of π) such that

$$M(\pi, y) = \pi\{\partial C(y)/\partial y\} + R(y), \tag{16}$$

and this allows us to rewrite (14) in the form

$$\dot{\varepsilon} = \overset{\bullet}{\overline{\{C(y)\pi\}}} + R(y)\dot{y}. \tag{17}$$

Further, since

$$R(y) = M(0, y), \tag{18}$$

we may conclude from (7) and (8) that

$$\overset{\bullet}{\overline{p(y)}} = R(y)\dot{y}; \tag{19}$$

hence

$$\dot{\varepsilon} = \overset{\bullet}{\overline{\{C(y)\pi\}}} + \overset{\bullet}{\overline{p(y)}}, \tag{20}$$

which gives the total strain-rate as the sum of elastic and permanent parts.

Let us agree to call a process **elastic** at a given time if, at that time,

$$\begin{aligned} &\pi < y, \\ &\text{or both } \pi = y \quad \text{and} \quad \dot{\pi} < \dot{y}. \end{aligned} \tag{21}$$

Using this definition, the flow rule (11) may be written in the equivalent form

$$\dot{\overline{p(y)}} \geqslant 0 \text{ always,}$$

$$\dot{\overline{p(y)}} = 0 \text{ whenever the process is elastic.} \tag{22}$$

To verify this equivalency, note first that, by (18) and assumption (9) part 2, $R(y) > 0$. We may therefore conclude from (19) that (22) is equivalent to the requirement that

$$\dot{y} \geqslant 0 \text{ always,}$$

$$\dot{y} = 0 \text{ whenever the process is elastic.} \tag{23}$$

Assume that (22) is satisfied. If $\pi < y$, then the process is elastic, so that, by (23), $\dot{y} = 0$. Alternatively, if $\pi = y$, by (2) either $\dot{\pi} < \dot{y}$ or $\dot{\pi} = \dot{y}$. In the former case the process is elastic, so that $\dot{\pi} < \dot{y} = 0$; in the latter case $\dot{\pi} = \dot{y} \geqslant 0$. Thus, in either case, $\dot{y} = \max\{\dot{\pi}, 0\}$. This establishes that (22) implies (11). Conversely, assume that (11) is satisfied. Then, trivially, $\dot{y} \geqslant 0$. Assume that the process is elastic. If $\pi < y$, then (11) yields $\dot{y} = 0$. If $\pi = y$, then $\dot{\pi} < \dot{y}$ and (11) again yields $\dot{y} = 0$. Thus (11) implies (23) and hence (22). Therefore (11) and (22) are equivalent.

Finally, we note that the definition $F(\pi, y) \equiv F(\pi, y_0)$ for $y \leqslant y_0$ extends the yield envelope to zero stress. With this definition the constitutive equation (1) is valid for $y \geqslant 0$, but the portion of the yield envelope $\varepsilon = Y(y)$ for $0 \leqslant y \leqslant y_0$ is the initial elastic curve. (Note that the yield envelope is then, in general, no longer smooth.)

1.2. *The viscoelastic portion of the model*

We introduce rate-dependence by regarding π as a **pseudo-stress** and relating it to the (true) **stress** σ by a **linear viscoelastic law**:

$$\pi(t) = \int_0^t J(t-s)\dot{\sigma}(s)\,\mathrm{d}s \tag{24}$$

with $J(t)$ the **creep function**. We assume that

$$J(t) \text{ is strictly increasing and uniformly bounded.} \tag{25}$$

In writing (24) we assume that

$$\sigma(t) = 0 \quad \text{for all} \quad t \leqslant 0, \tag{26}$$

consistent with (10).

2. THEORY WITH TEMPERATURE DEPENDENCE

2.1. *The plastic portion of the model*

First, we generalize (1) by writing

$$\varepsilon = F(\pi, y, T) \tag{27}$$

with T the **temperature** and ε the compressive strain with the strain due to thermal expansion removed.† For a test at constant temperature T, the yield stress y has the same interpretation as in the isothermal theory. Thus, in principle, the function F can be determined by isothermal tests alone. As before, the yield stress must be larger than an initial yield-stress

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† This assumption is quite strong as it requires the total strain be the superposition of ε and a strain due to thermal expansion.

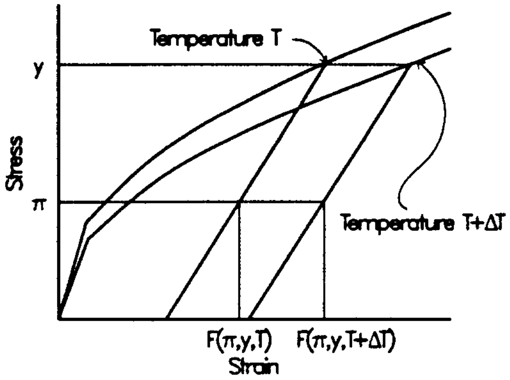

Fig. 3. Graphical description of the assumption $N(\pi, y, T) > 0$.

$y_0(T) \geqslant 0$, which we allow to depend on the temperature T; and, since ε does not include thermal expansion, we add the requirement that

$$F(0, y_0(T), T) = 0. \tag{28}$$

Of course, tacit in the relation (27) is the assumption that

$$\pi \leqslant y. \tag{29}$$

The **yield envelope** is now the surface in ε, y, T space given by

$$\varepsilon = Y(y, T), \tag{30}$$

where

$$Y(y, T) \equiv F(y, y, T) \tag{31}$$

and the **permanent strain** $p(y, T)$ now depends on temperature:

$$p(y, T) \equiv F(0, y, T). \tag{32}$$

We assume that the functions

$$C \equiv \partial F/\partial \pi, \quad M \equiv \partial F/\partial y, \quad N \equiv \partial F/\partial T \tag{33}$$

satisfy

$$C > 0, \quad M > 0, \quad N > 0, \tag{34}$$

with $C(y, T)$ independent of π, an assumption which renders the elastic loading and unloading curves linear under isothermal conditions. The physical meaning of (34) part 2 is as explained for the isothermal theory (cf. Fig. 2), while (34) part 3 is the requirement that the material soften with increasing temperature and yield more readily, as seen in Fig. 3.

Processes now consist of functions $\varepsilon(t)$, $\pi(t)$, $y(t)$, and $T(t)$ with [cf. (28)]

$$\pi(t) = 0, \quad T(t) = T_0, \quad y(t) = y_0(T_0), \quad \varepsilon(t) = 0 \quad \text{for} \quad t \leqslant 0. \tag{35}$$

The constant T_0 represents an initial temperature, which of course may vary from process to process.

As in the isothermal theory, we use the integrability conditions

$$\partial C/\partial y = \partial M/\partial \pi, \quad \partial C/\partial T = \partial N/\partial \pi \tag{36}$$

to establish the existence of functions $R(y, T)$ and $H(y, T)$ such that

$$\begin{aligned} M(\pi, y, T) &= \pi\{\partial C(y, T)/\partial y\} + R(y, T), \\ N(\pi, y, T) &= \pi\{\partial C(y, T)/\partial T\} + H(y, T), \end{aligned} \tag{37}$$

and then use these relations to write (27) in the form

$$\dot{\varepsilon} = \overset{\bullet}{\overline{\{C(y, T)\pi\}}} + R(y, T)\dot{y} + H(y, T)\dot{T}. \tag{38}$$

Since

$$R(y, T) = M(0, y, T), \quad H(y, T) = N(0, y, T), \tag{39}$$

(32) and (33) yield

$$\overset{\bullet}{\overline{p(y, T)}} = R(y, T)\dot{y} + H(y, T)\dot{T}, \tag{40}$$

and

$$\dot{\varepsilon} = \overset{\bullet}{\overline{\{C(y, T)\pi\}}} + \overset{\bullet}{\overline{p(y, T)}}. \tag{41}$$

Thus the strain rate is again the sum of elastic and permanent parts.

The chief difficulty in dealing with a varying temperature is specifying a suitable flow rule. Using (22) as a guide, we take as our **flow rule**

$$\begin{aligned} &\overset{\bullet}{\overline{p(y, T)}} \geqslant 0 \text{ always,} \\ &\overset{\bullet}{\overline{p(y, T)}} = 0 \text{ whenever the process is elastic.} \end{aligned} \tag{42}$$

The term **elastic** is still defined by (21):

$$\begin{aligned} &\pi < y, \\ &\text{or both } \pi = y \quad \text{and} \quad \dot{\pi} < \dot{y}. \end{aligned} \tag{43}$$

Further, we shall refer to a process as plastic (at a given time) if it is not elastic; that is, if

$$\pi = y \quad \text{and} \quad \dot{\pi} = \dot{y}. \tag{44}$$

Next, note that, by (34) and (39),

$$R(y, T) > 0, \quad H(y, T) > 0, \tag{45}$$

and

$$A(y, T) \equiv H(y, T)/R(y, T) \tag{46}$$

is defined. In view of (40) and (45), the rule (42) is equivalent to

$$\begin{aligned} &\dot{y} \geqslant -A(y, T)\dot{T} \quad \text{always,} \\ &\dot{y} = -A(y, T)\dot{T} \quad \text{whenever the process is elastic.} \end{aligned} \tag{47}$$

The equation $\varepsilon = Y(y, T)$, when solved for y as a function of ε and T, gives the dependence—during yielding—of the yield stress on strain and temperature. The result (47) shows that the yield stress can also change in the elastic range due to changing temperature; in fact, since $A > 0$, y decreases with increasing temperature.

The flow rule (42) has the equivalent form:

$$\dot{y} = \begin{cases} \max\{\dot{\pi}, -A(y, T)\dot{T}\} & \text{if } \pi = y, \\ -A(y, T)\dot{T} & \text{if } \pi < y, \end{cases} \tag{48}$$

which generalizes the isothermal flow-rule (11). The proof that (42) and (48) are equivalent is completely analogous to the proof, given in the paragraph following (22), of the corresponding equivalence in the isothermal theory. [Here (47) takes the place of (23).]

As before, the definition $F(\pi, y, T) = F(\pi, y_0(T), T)$ for $y \leqslant y_0(T)$ extends the yield envelope to zero stress; with this definition the portion of the yield envelope for $0 \leqslant y \leqslant y_0(T)$ is the initial elastic curve. We shall not use this extension in what follows.

Consider a process which is arbitrary up to a given time t_0, but which is elastic for times $t \geqslant t_0$. Let p_0 denote the permanent strain at time t_0. Then, trivially, $p(y(t), T(t)) = p_0$ for $t \geqslant t_0$ and, integrating (41), we find that

$$\varepsilon(t) = p_0 + C(y(t), T(t))\pi(t) \tag{49}$$

for $t \geqslant t_0$. As we shall see, this result has an interesting consequence within the complete theory including viscoelastic effects.

2.2. *The viscoelastic portion of the model*

This part of the model is changed in a straightforward manner: the pseudo-stress π is still related to the stress σ by a viscoelastic law, but we assume that the law is **thermorheologically simple** (cf., for example, Pipkin [2]). We thus assume there is a change in time scale $\xi = \xi(t)$—with $\mathrm{d}\xi/\mathrm{d}t$ a function of current temperature—such that the isothermal law (24) remains valid if we use reduced time ξ in place of real time t. More precisely, we assume there is a shift function $\phi(T)$ such that if

$$\xi(t) = \int_0^t \phi(T(s))\,\mathrm{d}s, \tag{50}$$

then

$$\pi(t) = \int_0^t J(\xi(t) - \xi(s))\dot{\sigma}(s)\,\mathrm{d}s, \tag{51}$$

with J the (isothermal) **creep function** corresponding to a base temperature T_0 at which $\phi(T_0) = 1$, and with J consistent with (25). The constitutive equations (50) and (51) give the pseudo-stress $\pi(t)$ at time t as a function of the histories of stress and temperature up to time t; for convenience, we write σ^t and T^t for these histories and

$$\pi(t) = \mathfrak{Z}(\sigma^t, T^t) \tag{52}$$

for the corresponding functional relation.

2.3. *The complete model*

In view of the discussion above, the complete model has the form

$$\varepsilon = F(\pi, y, T),$$

$$\dot{y} = \begin{cases} \max\{\dot{\pi}, -A(y, T)\dot{T}\} & \text{if } \pi = y, \\ -A(y, T)\dot{T} & \text{if } \pi < y, \end{cases}$$

$$\pi(t) = \mathfrak{J}(\sigma^t, T^t). \tag{53}$$

Processes now consist of functions $\varepsilon(t)$, $\pi(t)$, $y(t)$, $T(t)$, $\sigma(t)$ consistent with the constitutive equations (53) and the initial conditions

$$\sigma(t) = \varepsilon(t) = \pi(t) = 0, \quad T(t) = T_0, \quad y(t) = y_0(T_0) \quad \text{for} \quad t \leqslant 0. \tag{54}$$

Because of the form of (52) we shall generally consider the stress $\sigma(t)$ and temperature $T(t)$ as input and the strain $\varepsilon(t)$ as output, the pseudo-stress $\pi(t)$ and the yield stress $y(t)$ now being no more than intermediate calculations.

2.3.1. *Creep test.* Consider a creep test at a *constant* but arbitrary temperature T:

$$\sigma(t) = \sigma_0 \quad \text{for all} \quad t > 0. \tag{55}$$

Then, by (50) and (51),

$$\pi(t) = \sigma_0 J(\phi(T)t). \tag{56}$$

Case 1 (Constant compliance and nonzero initial yield-stress). Assume that the amplitude σ_0 is sufficiently small that $\sigma_0 J(t) \leqslant y_0(T)$ for all time, so that the process remains within the initial elastic-range. Then, by (49) with $p_0 = 0$,

$$\varepsilon(t)/\sigma_0 = CJ(\phi(T)t). \tag{57}$$

Further, it is clear that we may scale π so as to render

$$C = 1; \tag{58}$$

granted this is done, we find that

$$\varepsilon(t)/\sigma_0 = J(\phi(T)t), \tag{59}$$

which is the standard formula for the creep function.

Case 2 (General case). Consider now the general case in which the compliance is not necessarily constant, and in which the initial yield-stress may vanish. By (25) and (56), $\pi(t)$ is increasing and bounded. Further, (49) (with $p_0 = 0$) and (30) imply that, for σ_0 small enough,

$$\begin{aligned} \varepsilon(t) &= \sigma_0 C(y_0(T), T) J(\phi(T)t) \quad \text{if } y_0(T) > 0, \\ \varepsilon(t) &= Y(\sigma_0 J(\phi(T)t), T) \quad \text{if } y_0(T) = 0. \end{aligned} \tag{60}$$

Thus, since J is bounded, we have the following approximation for small stress σ_0:

$$\varepsilon(t)/\sigma_0 = \kappa(T) J(\phi(T)t) + O(\sigma_0), \tag{61}$$

where

$$\begin{aligned} \kappa(T) &= C(y_0(T), T) \quad \text{if} \quad y_0(T) > 0, \\ \kappa(T) &= \partial Y(y, T)/\partial y \quad \text{at} \quad y = 0 \quad \text{if} \quad y_0(T) = 0, \end{aligned} \tag{62}$$

and where the $O(\sigma_0)$ term may be dropped for $y(T_0) > 0$. Equation (61)—with the $O(\sigma_0)$ term neglected and the $\kappa(T)$ term scaled to unity—is again the standard relation for the creep function. In general, one cannot expect to scale $\kappa(T)$ to unity at all temperatures. However, if we assume that the material is thermorheologically simple† for creep tests at sufficiently small amplitude, then we are led to the conclusion that

$$\kappa(T) = 1 \quad \text{for all} \quad T, \tag{63}$$

and hence that

$$\varepsilon(t)/\sigma_0 = J(\phi(T)t) + O(\sigma_0). \tag{64}$$

Henceforth we take (63) *as an additional hypothesis.*

2.3.2. *General behavior within the elastic range.* Consider now a process in which the material undergoes an arbitrary loading history, possibly involving yielding, and then (for $t \geqslant t_0$) has pseudo-stress $\pi(t)$ within the elastic range:

$$\pi(t) < y(t) \quad \text{for} \quad t \geqslant t_0. \tag{65}$$

Let p_0 denote the plastic strain at t_0. Then (49) and (53) yield

$$\varepsilon(t) - p_0 = C(y(t), T(t))\,\mathfrak{J}(\sigma', T') \tag{66}$$

for $t \geqslant t_0$. Thus the strain, measured relative to the permanent strain, is given by the usual thermorheologically simple relation $\mathfrak{J}(\sigma', T')$ multiplied by a **correction factor**‡ $C(y(t), T(t))$ which accounts for past yielding and temperature. If we assume that the elastic behavior—relative to the permanent set and for a given value of yield stress—is thermorheologically simple, then the compliance should be taken independent of T,

$$C(y, T) = C(y). \tag{67}$$

Given this, (66) reduces to

$$\varepsilon(t) - p_0 = C(y(t))\,\mathfrak{J}(\sigma', T'). \tag{68}$$

The current yield stress is an indication of the amount of internal damage to the material during its past history, and for that reason we expect $C(y)$ to increase with increasing y. If this is so, then (68) asserts that *the behavior after yielding remains thermorheologically simple, but with stiffness appropriately reduced to account for past damage.*

2.3.3. *Behavior at transitions.* We now study the behavior of processes at elastic–plastic and plastic–elastic transitions. What is interesting here is the possibility that a process generated by *smooth* inputs $\sigma(t)$ and $T(t)$ will have a *nonsmooth* output $\varepsilon(t)$ due to a possible discontinuity in the yield stress $y(t)$ at the transition.

Consider a transition which occurs at time τ, and for any function $\psi(t)$ let $[\psi]$ denote the *jump* in ψ across the transition:

$$[\psi] = \psi(\tau^+) - \psi(\tau^-). \tag{69}$$

As mentioned above, we will limit our discussion to transitions generated by smooth inputs,

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† In the sense that the reduced time accounts for all temperature effects other than the dependence of yield stress on temperature.

‡ Recall that our normalization ensures that $C(0, T) = 1$.

so that

$$[\dot{\sigma}] = [\dot{T}] = 0. \tag{70}$$

Thus, in view of (50) and (51),

$$[\dot{\pi}] = 0; \tag{71}$$

hence (40) and (41) yield

$$[\dot{\varepsilon}] = \{\pi(\partial C/\partial y)+R\}[\dot{y}]. \tag{72}$$

Consider first an elastic–plastic transition. Suppose that a process which had been elastic becomes plastic at time τ. Then $\dot{y} = \dot{\pi}$ at τ^+, while $\dot{y} = -A\dot{T}$ at τ^-; thus, since π and T are smooth, $[\dot{y}] = \dot{\pi}+A\dot{T}$ at time τ. Moreover, the only restriction on $\dot{\pi}$ and $\dot{T}$ at τ is that $\dot{\pi} \geqslant -A\dot{T}$. Thus in general $[\dot{y}] \neq 0$ and we may conclude that $[\dot{\varepsilon}]$ is generally nonzero. Consider next a plastic–elastic transition: at time τ the process, which had been plastic, becomes elastic. Then the argument given above implies that $[\dot{y}] = -\dot{\pi}-A\dot{T}$, but here, because of the yield condition (48), $\dot{\pi} = -A\dot{T}$ at τ and $[\dot{\varepsilon}] = 0$. Thus we have the following interesting result for a transition at which the input stress and temperature are smooth. *If the transition is elastic-plastic, then the strain-rate $\dot{\varepsilon}$ generally has a jump discontinuity; if the transition is plastic–elastic, then $\dot{\varepsilon}$ is continuous.* (73)

3. CHARACTERIZATION

Here we describe, in general terms, a procedure for characterizing the response of a material presumably modeled by this constitutive theory.

3.1. *Determination of the creep function*

First consider creep tests at several *constant* temperatures. We assume that the stress level σ_0 is sufficiently low that (64), with the $O(\sigma_0)$ term neglected, is valid: for each T,

$$J(\phi(T)t) = \varepsilon(t)/\sigma_0. \tag{74}$$

Since

$$\log\,(\phi(T)t) = \log\,t+\log\,(\phi(T)), \tag{75}$$

if ε/σ_0 is plotted against $\log\,t$ for each such creep test, then by translating the curves by an amount $\log\,(\phi(T))$ the curves should coincide; we can use this standard procedure for determining the shift function $\phi(T)$.

The creep function J and the shift function ϕ must be available to use for the remaining characterization procedure.

3.2. *Determination of the initial elastic-curve and the yield envelope*

Experiments involving monotonic loading (ramp tests) at constant temperature now suffice to determine the initial elastic curve $\varepsilon = F(\pi, y_0, T)$ and the yield envelope $\varepsilon = Y(\pi, T)$. Given any such test, one may use the creep law (51), characterized in Section 3.1, in conjunction with the experimental $\sigma(t)$ to determine $\pi(t)$. Since $\sigma(t)$ is monotone, so is $\pi(t)$. Thus a plot of $\pi(t)$ versus the experimental $\varepsilon(t)$ reveals the form of $F(\cdot, y_0, T)$ and $Y(\cdot, T)$ at the given temperature. In principle a single experiment suffices to determine these functions at that temperature, but a useful test of the constitutive hypothesis that the creep law encompasses the entire rate dependence is afforded by taking a series of such tests at differing rates and comparing the corresponding graphs.

Because ramp tests are often inaccurate for times near the starting time of the experiment, and hence for very low stress levels, the condition (63) should be a part of the characterization procedure.

Finally, to determine the dependence of $F(\cdot, Y_0, T)$ and $Y(\cdot, T)$ on T, this procedure is repeated at different temperatures.

3.3. *Determination of the subsequent elastic curves*

To determine $F(\pi, y, T)$ for $\pi < y$ it is necessary to have constant-temperature experiments involving unloading, and, optimally, reloading. For example, loadings such as those shown in Fig. 10 could be used but usually single-rate, dwell-less sawtooth tests with a series of increasing peak levels are used for this purpose. From the experimental $\sigma(t)$ one computes $\pi(t)$ using (51) and then considers a π–ε plot over each interval on which y is constant and $\pi < y$. The corresponding π–ε plots, each for a different value of y, may be used to construct $F(\pi, y, T)$ at each fixed value of T. This procedure is then repeated at different temperatures to determine the dependence on T.

4. TEST MATERIALS AND METHODS

The material studied is PBX-9502, a plastic-bonded explosive. The material is 95% triaminotrinitrobenzine (TATB), an organic explosive in the form of fine crystals, bound together with 5% Kel-F 800, a plastic, namely poly(trifluorochloroethylene).

Specimens were 41.3 mm dia. by 50.8 mm long cylinders usually instrumented with both bonded strain gages and clip-on extensometers. The relatively low length-to-diameter ratio suggests a possible problem with end-effects. However the bonded strain gages were located on the specimens at the mid-point, and we used an elastic finite-element analysis to deduce that the measured strains would not be seriously affected by the lateral frictional restraints on the ends of the specimen, except near failure. Failure loads and strains near failure condition could of course be greatly influenced by the end conditions. The strain gages were 1/4 in. gage length biaxial gages mounted on opposite sides of the specimen. Extensometers with 25 mm gage length were mounted straddling the strain gages. The extensometers tend to exhibit hysterises under reverse loading conditions, an effect that became apparent when comparing the extensometer and strain gage results. Because of the hysterises and the enhanced end effects with the extensometers, we customarily used the strain gage results. In fact, all creep data were obtained using only bonded strain gages because of space limitations in the creep apparatus.

The constant-rate-to-failure and cylic-load tests were all performed on a screw-driven, servo-controlled, constant-rate testing machine. Early creep tests, see Peeters[3], were done using a lever-arm compression tester, while later tests were done in dead-weight loading frames using a compression cage.

A data acquisition system driven by a mini-computer was used to control the application of pulsed excitation to the strain gages and then to read the bridge output signals. The voltage readings from the bridge were stored on a floppy diskette before being transferred to a general purpose computer for conversion to strain values and correction for thermal effects and finally for display and manipulation as described in the next section.

5. CONSTRUCTION AND TESTING OF A MODEL

In [1] we discussed the problem of modeling the filled polymer PBX-9502, among other materials, with the isothermal model. As observed there, the model gives good predictions for the material, with some shortcomings (cf. the figures below). First, there appeared to be a small amount of rate-dependence not accounted for by the viscoelastic superposition; second, residual strains were not very precisely predicted; and third, the assumption of linearity in the viscoelastic portion of the model is not fulfilled, as can be seen in Fig. 12. The predictions for creep at a higher level of stress than the characterizing level are the least accurate among all of the tests which were analyzed. Overall, though, it was felt that the accuracy of the model was sufficient for most purposes. In approaching the valuation of the model for varying temperatures, these inherent problems with the isothermal model must be kept in mind.

The data available was in the form of experimental data $\sigma(t)$, $\varepsilon(t)$ for several experiments at each of three temperatures. The experimental data was smoothed with spline fits, with errors much less than the point-to-point variation of the data. The chief errors in the creation of the model thus were caused by experimental variability. In particular, the material showed large variations in properties from sample to sample, and the problems involved in safely machining and testing the specimens guaranteed that the tests could not be repeated sufficiently frequently to make a precise evaluation of these variations. Also the temperature control in the testing was difficult, and some significant variation in experimental values could be attributed to this.† Finally, there was an initialization problem at the beginning of each test, which produced some scatter of values in the early datapoints.

The main source of evaluation errors was the inaccuracy, mentioned above, involved in using creep data to approximate J.

For each of the three temperatures used we obtained creep data, at stress levels of 1.0 or 1.87 MPa, and used these to create a creep function, using a piece-wise linear fit to the log strain versus log time plots.‡ A quite reliable fit over an experimental time period of 25,000 minutes was obtained by choosing a partition which corresponds in real time to the intervals (0, 12) and (12, 25000). Figure 4 shows the three creep functions plotted as log J versus log t. It is clear that temperature–time superposition will not be precise for these models. The values of the log J/log t slopes would be equal if this were to be the case; here they differ by as much as 55%.§ Indeed, one might expect that this simple model of temperature variability will seldom be precisely true. (The choice of using temperature–time superposition in the model even though it clearly is not precise was dictated by the simplicity and computing ease which it guarantees.) Rather than try to shift the experimental data and do a single fitting of a master curve, it was decided that it would be equally accurate and more systematic to use these three log–log curves to construct a master curve. Based on the belief that the creep measurements are (experimentally) most accurate in the higher ranges it was decided to concatenate the three graphs by successively shifting the breakpoint of the higher temperature curves to the right to intersect the master graph, and then replacing the part of the master graph which extends beyond the intersection by the upper part of the shifted curve. The resulting master curve is shown, with all three isothermal curves, in Fig. 4 and the shifted-master curves are contrasted with the isothermal curves in Fig. 5. It is clear that the resulting master curve is a compromise which favors high temperature predictions at longer times and low-temperature predictions at shorter times. Our choice here should be adjusted to the expected demands of the use of the model. If the model is to be used mostly in applications which involve long time dwells at low temperatures one should make a different choice and favor the low temperature creep function for the ultimate continuation. The virtue of this method of fitting over fitting of shifted multiple temperature data is that it allows a clear-cut choice, as described, and thus allows one to predict the direction of error in applications.

Using the appropriately shifted master creep function, the virgin curve for each temperature was determined by the data from a single ramp test at a medium rate (among a choice of three rates). The resulting curves, pictured in Fig. 6, differed moderately and, surprisingly, the three did not show a monotone behavior with temperature. Care was taken in the construction of the curves to ensure that the slope was unity in a neighborhood of

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† Compensation for thermal expansion removed part of the strain error due to this variation, but of course the constitutive equations are quite sensitive to temperature.

‡ This 'power-law' format was chosen because it gives a better-behaved extrapolation beyond the fitting interval than other choices.

§ More precisely: if one took the mean as the master value, the 70°C value would be 35% greater than the mean value.

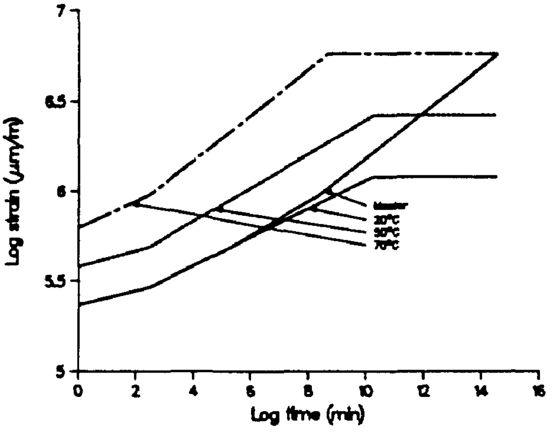

Fig. 4. Creep functions at three temperatures and master creep curve.

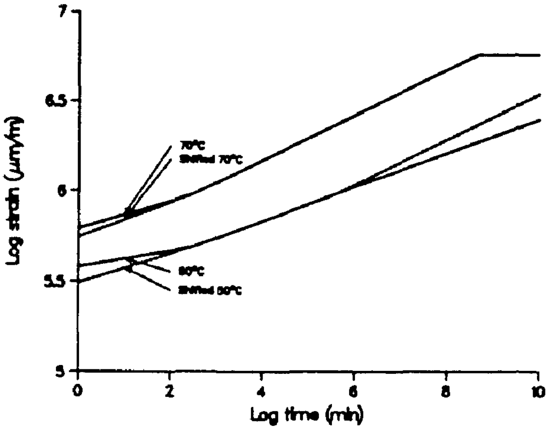

Fig. 5. True creep vs shifted master creep curves.

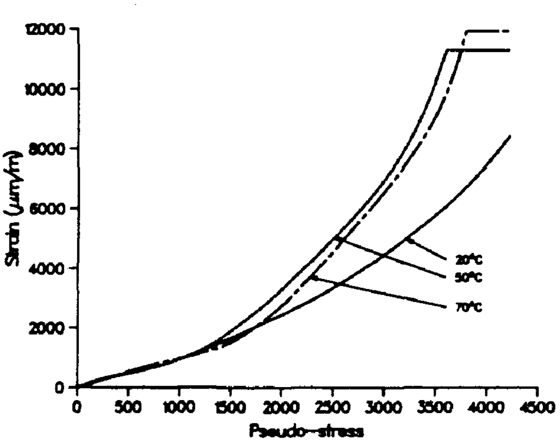

Fig. 6. Virgin response curves for PBX-9502.

the origin, as is required by the theory. In fact, the data itself did not justify this adjustment; however experimental errors at early times makes the data suspect.

For each temperature, the compliance in the damaged zone was then determined by examining the data from a single test, having an increasing-amplitude-sawtooth stress time history. This yielded in each case a set of strain vs pseudo-stress points; a curve was fit to

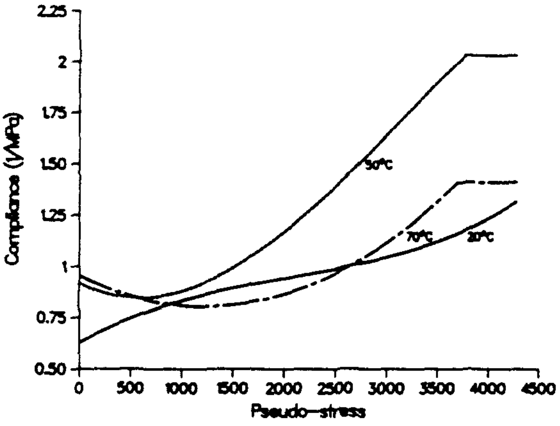

Fig. 7. Compliances for damaged material PBX-9502.

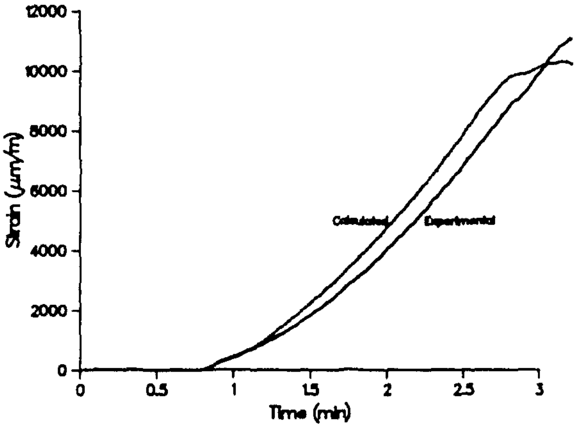

Fig. 8. Typical different-rate verification tests, 20 C.

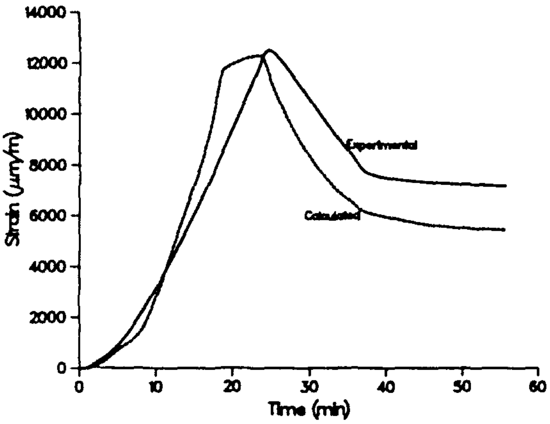

Fig. 9. Typical different-rate verification tests, 70 C.

this data for each temperature, with the results shown in Fig. 7. There was no common pattern to the three curves and they are not monotone in the temperature.

The models constructed at each temperature were evaluated by comparing their predictions to other tests at the same temperature. The results of some of these evaluations are shown in Figs 8–12.

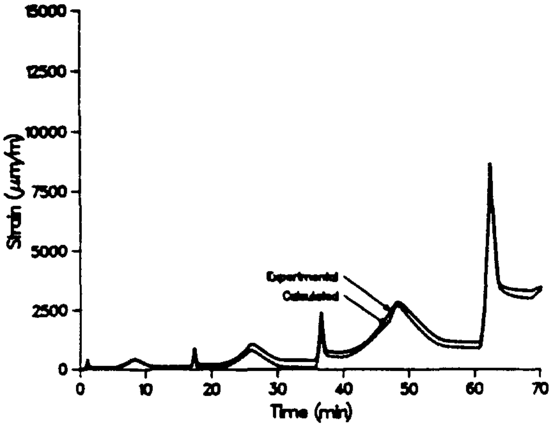

Fig. 10. Typical multi-rate predictions for PBX-9502, 60 C.

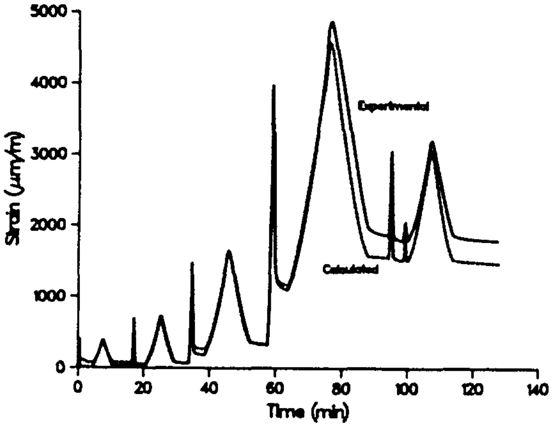

Fig. 11. Typical multi-rate predictions for PBX-9502, 20 C.

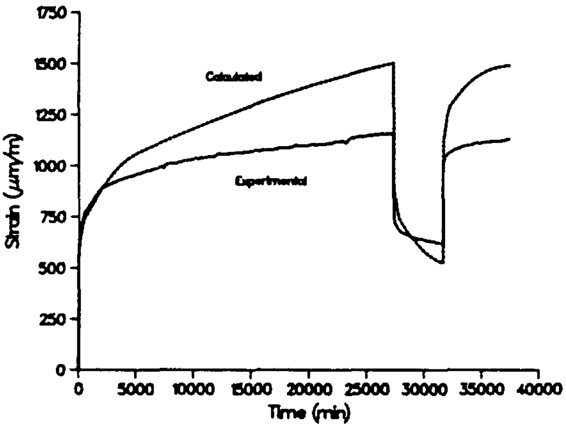

Fig. 12. Creep and recovery test for PBX-9502, 50 C.

Figures 8 and 9 show the prediction of the model for simple ramp tests which are run at rates which are ten times the rate of the characterizing ramp test. The predictions are not exact, indicating a residual rate-effect, but they were felt to be acceptable. In Figs 10 and 11 the results for some multi-rate tests are shown ; the agreement is remarkable. Figure 12 illustrates the prediction for a combined creep-stress relaxation test. The test was run at

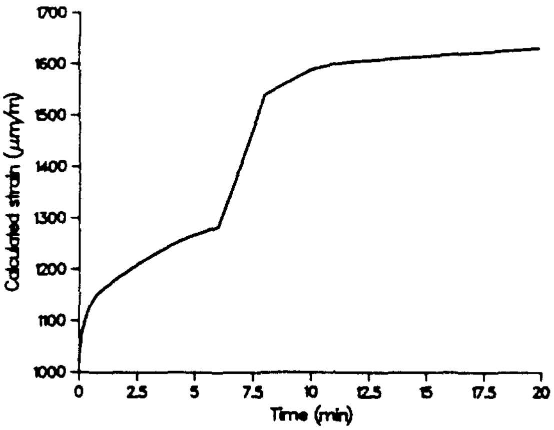

Fig. 13. Prediction for temperature-cycling creep test.

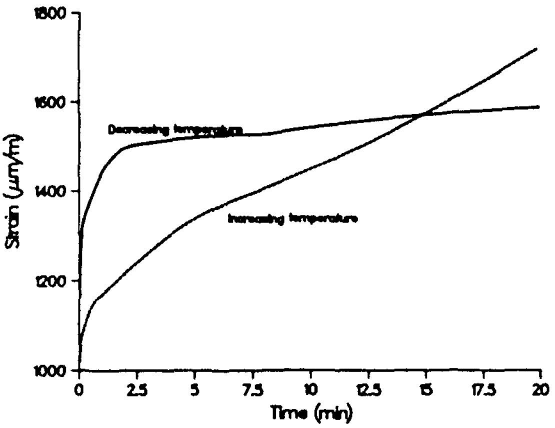

Fig. 14. Predicted creep with increasing and decreasing temperature.

50°C, the temperature at which the shifted creep function was the least accurate, which by itself introduces inaccuracy, but the large discrepancy illustrates that the inaccuracies will be compounded for very long time intervals.

6. REMARK ON TEMPERATURE-VARYING TESTS

It was not possible to obtain sufficient temperature-varying tests for the material in question to obtain useful comparisons. There are, however, some interesting effects which are predicted by the theory. The more interesting results occur in creep experiments; for example, Fig. 13 shows the model's prediction for a creep test during which the temperature is increased from a low to a high value, held at that value, and then reduced back to the low value. Initially the material is being damaged (the stress–strain point is on the yield envelope) and the strain increases in a typical creep pattern. As the temperature lowers, the yield envelope lowers as well and, at a certain point, the stress–strain point falls below the current yield envelope. Thereafter the material behaves elastically until the time at which the strain increases sufficiently that the stress–strain point is on the new yield envelope (corresponding to the current, highest, temperature). Then as the temperature is reduced, the material continues to yield at a varying rate as the yield envelope changes with the temperature. Similarly, Fig. 14 illustrates the model's predictions for creep with temperatures increasing and decreasing slowly enough that the material is always being damaged.

7. CONCLUSIONS

The multi-temperature model was fit to the material with moderate success. The single-temperature fitting was quite successful, but the material appears not to obey closely the assumed temperature–time superposition rule. An approximate master creep curve led to reasonable, but not excellent predictions for tests at various temperatures. Nonetheless, the overall simplicity of the model and the ease of systematic and verifiable fitting makes it a reasonable model for this material.†

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REFERENCES

1. R. V. Browning, M. E. Gurtin and W. O. Williams, A one-dimensional viscoplastic constitutive theory for filled polymers. *Int. J. Solids Structures* **20**, 921–934 (1984).
2. A. C. Pipkin, *Lectures on Viscoelasticity Theory*, Springer, Berlin (1972).
3. R. L. Peeters and R. M. Hackett, Constitutive modelling of plastic-bonded explosives. *Exp. Mech.* **21**, 111–116 (1981).

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† Tests on some of the other materials considered in [1] indicate that, for those materials, the time-temperature superposition rule is much more consistent. For example, for the material referred to as 900-10 in [1], creep functions have been obtained at 10, 20, 30, and 40 C and it is found that the disagreement between the slopes of the log j vs log t lines is only 12% at the extreme temperatures.